ANNALS OF MATHEMATICS.

ORMOND STONE, Editor.

WILLIAM M. THORNTON, Associate Editor.
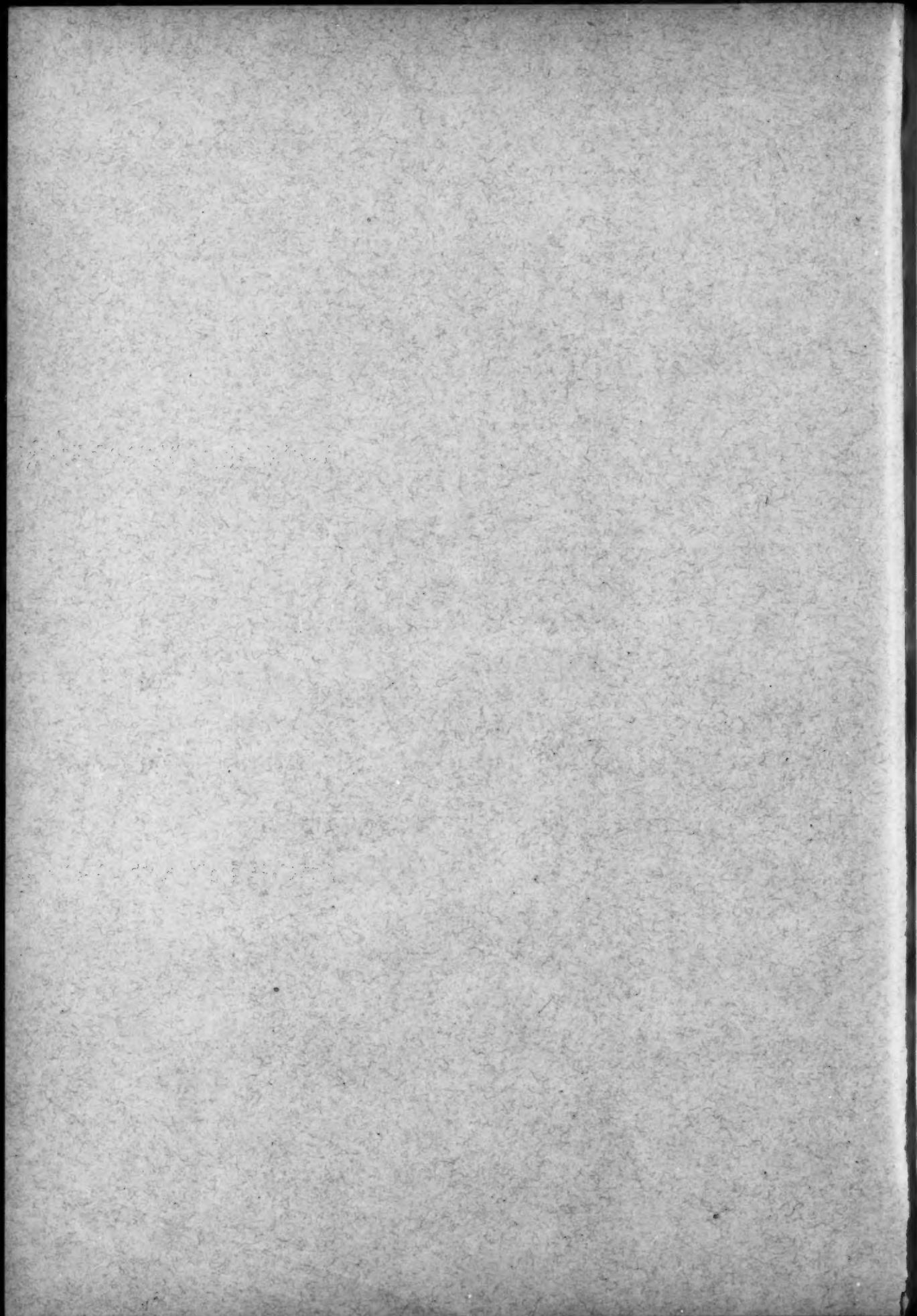
A SOLUTION OF KEPLER'S PROBLEM FOR PLANETARY ORBITS OF HIGH ECCENTRICITY.

By DR. H. A. HOWE, Denver, Col.

The solution of Kepler's Problem involves the transcendental equation

$$M = E - e \sin E, \tag{1}$$

in which M and E are respectively the mean and eccentric anomalies, while e is the eccentricity.

Let
$$\begin{aligned} F(E) &= \tfrac{1}{2}\left[E - M - \sin(E - M) \right] \\ &= \tfrac{1}{12}\sin^3(E - M) + \tfrac{3}{80}\sin^5(E - M) \\ &\quad + \tfrac{5}{224}\sin^7(E - M) + \ldots . \end{aligned} \tag{2}$$

From (1) and (2) we easily get Grunert's equation,

$$\frac{1 + e}{1 - e} = \frac{\cos \tfrac{1}{2}M \sin(E - \tfrac{1}{2}M) + F(E)}{\sin \tfrac{1}{2}M \cos(E - \tfrac{1}{2}M) - F(E)}. \tag{3}$$

Equation (3) gives

$$\tan(E - \tfrac{1}{2}M) = \frac{1 + e}{1 - e}\tan \tfrac{1}{2}M - \frac{2F(E)}{(1 - e)\cos \tfrac{1}{2}M \cos(E - \tfrac{1}{2}M)}. \tag{4}$$

Assume that
$$\tan(E' - \tfrac{1}{2}M) = \frac{1 + e}{1 - e}\tan \tfrac{1}{2}M. \tag{5}$$

Substituting in (4) according to (5) and reducing, we obtain

$$\sin(E' - E) = \frac{\cos(E' - \tfrac{1}{2}M)}{(1 - e)\cos \tfrac{1}{2}M} \cdot 2F(E) \tag{6}$$

$$= k \cdot 2F(E). \tag{7}$$

Hence for moderate eccentricities, $E' - E$ is approximately equal to $2F(E)$.

Let
$$2\eta = E' - M - \sin(E' - M). \tag{8}$$

Then $E' - E$ is likewise nearly equal to 2η, and $E' - M - 2\eta$ is nearly equal

to $E - M$. From (2) we obtain by differentiation

$$dF(E) = \left[\tfrac{1}{4} \sin^2(E - M) + \tfrac{3}{16} \sin^4(E - M) + \; . \; . \; . \right] \cos(E - M) d(E - M). \quad (9)$$

Equation (9) shows that a small change in the value of $E - M$ produces a much smaller change in $F(E)$; therefore, we write with but slight error

$$2 F(E) = E' - M - 2\eta - \sin(E' - M - 2\eta). \quad (10)$$

From (8), omitting terms of the seventh and higher orders, we have

$$\eta = \tfrac{1}{12} \sin^3(E' - M) + \tfrac{3}{80} \sin^5(E' - M). \quad (11)$$

Suppose that E' is changed by an amount equal to 2η; i. e. that $dE' = 2\eta$. Differentiation of (11) and reduction of the resulting equation, neglecting powers of $\sin(E' - M)$ higher than the fifth, gives

$$d\eta = \tfrac{1}{24} \sin^5(E' - M) \cos(E' - M); \quad (12)$$

$$\therefore \; \eta - d\eta = \tfrac{1}{12} \sin^3(E' - M) + \tfrac{3}{80} \sin^5(E' - M) - \tfrac{1}{24} \sin^5(E' - M) \cos(E' - M). \quad (13)$$

The last two terms of (13) have nearly the same magnitude, and opposite signs. We therefore write $\qquad \eta - d\eta = \tfrac{1}{12} \sin^3(E' - M). \quad (14)$

Now since (12) is obtained on the assumption that $dE' = 2\eta$,

$$\eta - d\eta = \tfrac{1}{2} \left[E' - M - 2\eta - \sin(E' - M - 2\eta) \right].$$

Hence by (10) and (14) we may write

$$F(E) = \tfrac{1}{12} \sin^3(E' - M). \quad (15)$$

Equations (15) and (7) give

$$\sin(E' - E) = \tfrac{1}{6} k \sin^3(E' - M). \quad (16)$$

Let $E' - M - \tfrac{1}{6} k \sin^3(E' - M)$ be an approximate value of $E - M$; then $2F\left[E' - M - \tfrac{1}{6} k \sin^3(E' - M)\right]$ will be the corresponding value of $2F(E)$, and by (7), $k \cdot 2F\left[E' - M - \tfrac{1}{6} k \sin^3(E' - M)\right]$ will be the corresponding value of $\sin(E' - E)$. Differentiation of (7) gives

$$\cos(E' - E) d(E' - E)$$
$$= k \left[\tfrac{1}{2} \sin^2(E - M) + \tfrac{3}{8} \sin^4(E - M) + \; . \; . \; . \right] \cos(E - M) d(E - M)$$
$$= k \left[\tfrac{1}{2} \sin^2(E - M) + \tfrac{1}{8} \sin^4(E - M) + \; . \; . \; . \right] d(E - M)$$
$$= k v \, d(E - M). \quad (17)$$

If $E - M$ be increased by $E' - E - \tfrac{1}{6} k \sin^3(E' - M)$, and the corresponding increment of $E' - E$ be denoted by $\Delta(E' - E)$, we have

$$\Delta(E' - E) = k v \left[E' - E - \tfrac{1}{6} k \sin^3(E' - M)\right] \sec(E' - E). \quad (18)$$

But $\qquad E' - E + \Delta(E' - E) = k \cdot 2F\left[E' - M - \tfrac{1}{6} k \sin^3(E' - M)\right]$
$$+ 2F(E' - E). \quad (19)$$

Substitute from (18) in (19), and add and subtract $\frac{1}{6}k \sin^3(E'-M)$ in the second member of the resulting equation; reduction gives

$$(E'-E)\left[1 + kv \sec(E'-E)\right] = \left[1 + kv \sec(E'-E)\right] \cdot \frac{1}{6}k \sin^3(E'-M)$$
$$+ k \cdot 2F\left[E'-M - \frac{1}{6}k \sin^3(E'-M)\right]$$
$$- \frac{1}{6}k \sin^3(E'-M) + 2F(E'-E). \qquad (20)$$

Division of (20) by the coefficient of $E'-E$, and subtraction of $2F(E'-E)$ from each side, gives

$$\sin(E'-E) = \frac{1}{6}k \sin^3(E'-M)$$
$$+ \frac{k\left\{2F\left[E'-M - \frac{1}{6}k \sin^2(E'-M)\right] - \frac{1}{6}\sin^3(E'-M)\right\}}{1 + kv \sec(E'-E)}$$
$$- \frac{kv \sec(E'-E) \cdot 2F(E'-E)}{1 + kv \sec(E'-E)}. \qquad (21)$$

But we have, with sufficient accuracy,

$$2F(E'-E) = \frac{1}{6}\sin^3(E'-E) = \frac{1}{1296}k^3 \sin^9(E'-M),$$

and
$$\frac{kv \sec(E'-E) \cdot 2F(E'-E)}{1 + kv \sec(E'-E)} = \frac{k^4 \sin^{11}(E'-M)}{2592(1 + kv)}. \qquad (22)$$

We next determine the error of (21) when its last term is neglected and the quantity $\sec(E'-E)$ is dropped from the denominator of the second fraction. The eccentricities of the orbits of all but 19 of the first 268 asteroids are below sin 15°. Of these 19, 15 lie between sin 15° and sin 20°, and 4 are greater than sin 20°; the orbit of Aethra (132) has the greatest eccentricity, e being equal to sin 22°.5. On the assumption that $e = \sin 23°$, we find that the maximum value of the last term of (21), as computed by (22), is 0″.003 +. If the factor $\sec(E'-E)$ in the denominator of the second fraction of (21) be dropped, an error of 0″.0002 may result. To compute v in the same denominator, we substitute for $E-M$ (in the expression given for v in (17)) the value $E'-M - \frac{1}{6}k \sin^3(E'-M)$. From equations (17)–(21) we see that, had $E'-E - \frac{1}{6}k \sin^3(E'-M)$, the increment of $E-M$, been *infinitesimal*, $E'-M - \frac{1}{6}k \sin^3(E'-M)$ might have been used for $E-M$ in computing v, without error. But since $E'-E - \frac{1}{6}k \sin^3(E'-M)$ is a small finite quantity, the proper value of $E-M$ to employ is

$$E'-M - \frac{1}{6}k \sin^3(E'-M) - \frac{1}{2}\left[E'-E - \frac{1}{6}k \sin^3(E'-M)\right].$$

The greatest error in $E'-E$ caused by using $E'-M - \frac{1}{6}k \sin^3(E'-M)$ for $E-M$, in finding v, is less than 0″.002. Greater accuracy is unnecessary, but could be attained, since $1 + kv$ is always close to unity, by using

$$E'-M - \frac{1}{6}k \sin^3(E'-M)$$
$$- \frac{1}{2}k\left\{2F\left[E'-M - \frac{1}{6}k \sin^3(E'-M)\right] - \frac{1}{6}\sin^3(E'-M)\right\},$$

or $\quad E' - M - \tfrac{1}{12}k \sin^3(E' - M) - k F[E' - M - \tfrac{1}{6}k \sin^3(E' - M)]$.

Equation (21) may therefore be written

$$\sin(E' - E) = \tfrac{1}{6}k \sin^3(E' - M)$$
$$+ \frac{k\{2F[E' - M - \tfrac{1}{6}k \sin^3(E' - M)] - \tfrac{1}{6}\sin^3(E' - M)\}}{1 + kv}. \quad (23)$$

When $e = \sin 23°$, the maximum numerical value of the last fraction of (23) is $38.''6$. A simple approximate value of this fraction in terms of k and $\sin(E' - M)$ may be obtained as follows:—

From (9) and (2), by putting $E' - M$ for $E - M$, $\tfrac{1}{6}k \sin^3(E' - M)$ for $d(E - M)$, and $1 - \tfrac{1}{2}\sin^2(E' - M)$ for $\cos(E' - M)$, we get

$$2F[E' - M - \tfrac{1}{6}k \sin^3(E' - M)] = \tfrac{1}{6}\sin^3(E' - M) + (\tfrac{3}{40} - \tfrac{1}{12}k)\sin^5(E' - M)$$
$$+ (\tfrac{5}{112} - \tfrac{1}{48}k)\sin^7(E' - M). \quad (24)$$

Since kv is small, we assume that

$$\frac{1}{1 + kv} = 1 - kv = 1 - \tfrac{1}{2}k \sin^2(E' - M). \quad (25)$$

Substitution of (24) and (25) in the last fraction of (23) gives for its value

$$\tfrac{1}{120}(9k - 10k^2)\sin^5(E' - M) + [\tfrac{1}{336}(15k - 7k^2) - \tfrac{1}{240}(9k^2 - 10k^3)]\sin^7(E' - M)$$

Since $\sin(E' - M) = e \sin E'$, neglecting powers higher than the fifth, the preceding expression becomes $\tfrac{1}{120}(9k - 10k^2) e^5 \sin^5 E'$. Because this involves the fifth power of e, its value is small for moderate eccentricities; nevertheless, the expression is not sufficiently accurate to be employed in seven-place computations with large eccentricities. As the second member of (23) is not easy to compute, it may be simplified, and its errors tabulated. It is not difficult to show that the maximum value of k is $(1 - e)^{-1}$, which is reached when $v = 0$. The maximum of kv is near $\tfrac{1}{2}e^2$, which is only 0.08 when $e = \sin 23°$; kv being, therefore, always small, neglect of the denominator of the last term of (23) introduces into $E' - E$ an error the value of which is not far from

$$\tfrac{1}{120} kv (9k - 10k^2) e^5 \sin^5 E', \text{ or } \tfrac{1}{240}(9k^2 - 10k^3) e^7 \sin^7 E'.$$

When $e = \sin 23°$ this error may amount to $3''$; when $e = \sin 15°$ it does not reach $0.''2$. Rejecting $1 + kv$ and writing $E' - E$ for $\sin(E' - E)$, (23) becomes

$$E' - E = k \cdot 2F[E' - M - \tfrac{1}{6}k \sin^3(E' - M)] \quad (26)$$

To expedite computation, two tables are needed, one giving the correction of the second member of (26), the other containing $2F(x)$ with the argument x. The computer having e and M given, would use (5) and (26), taking into account the tabulated correction to the value of $E' - E$ given by (26). From the value of E thus found, the true anomaly could be obtained by one of the usual methods.

EXTENSION OF ROLLE'S THEOREM.

By Mr. J. F. McCulloch, Ann Arbor, Mich.

If the $n + 1$ terms of any algebraic equation arranged in order of powers of x be multiplied respectively by any $n + 1$ terms of any series in arithmetical progression, the resulting equation has an odd number of real roots between every two adjacent positive or negative real roots of the equation so operated upon.

Let
$$f(x) \equiv p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \quad \ldots + p_{n-1} x + p_n = 0$$

be any algebraic equation, and $a, a + d, a + 2d, \ldots a + nd$ be any $n + 1$ terms of any arithmetical progression. After multiplication as specified above, let the resulting function of x be denoted by $(a, d) f_1(x)$.

Then
$$(a, d) f_1(x) \equiv a p_0 x^n + (a + d) p_1 x^{n-1}$$
$$+ (a + 2d) p_2 x^{n-2} + \ldots + (a + nd) p_n.$$

Let the roots of $f(x) = 0$ be $r_1, r_2, r_3, \ldots r_n$.

Then
$$f(x) = p_0 (x - r_1)(x - r_2)(x - r_3) \ldots (x - r_n).$$

Assume $f(x, y) \equiv p_0 x^n y^a + p_1 x^{n-1} y^{a+d} + p_2 x^{n-2} y^{a+2d} + \ldots + p_n y^{a+nd}.$

Then
$$f(x, y) \equiv p_0 y^a (x - r_1 y^d)(x - r_2 y^d)(x - r_3 y^d) \ldots (x - r_n y^d),$$

and
$$\frac{d}{dy} f(x, y) \equiv a p_0 x^n y^{a-1} + (a + d) p_1 x^{n-1} y^{a+d-1}$$
$$+ (a + 2d) p_2 x^{n-2} y^{a+2d-1} + \ldots + (a + nd) p_n y^{a+nd-1}$$
$$\equiv a p_0 y^{a-1} (x - r_1 y^d)(x - r_2 y^d)(x - r_3 y^d) \ldots (x - r_n y^d)$$
$$- d p_0 r_1 y^{a+d-1} (x - r_2 y^d)(x - r_3 y^d) \ldots (x - r_n y^d)$$
$$- d p_0 r_2 y^{a+d-1} (x - r_1 y^d)(x - r_3 y^d) \ldots (x - r_n y^d)$$
$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$
$$- d p_0 r_n y^{a+d-1} (x - r_1 y^d)(x - r_2 y^d) \ldots (x - r_{n-1} y^d).$$

If we make $y = 1$, $f(x, y)$ becomes $f(x)$, and $\frac{d}{dy} f(x, y)$ becomes $(a, d) f_1(x)$,

as we see from the first form of $\frac{d}{dy} f(x, y)$. Hence, from the second form,

$$(a, d) f_1(x) \equiv a p_0 (x - r_1)(x - r_2)(x - r_3) \ldots (x - r_n)$$
$$- d p_0 r_1 (x - r_2)(x - r_3) \ldots (x - r_n)$$
$$- d p_0 r_2 (x - r_1)(x - r_3) \ldots (x - r_n)$$
$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots$$
$$- d p_0 r_n (x - r_1)(x - r_2) \ldots (x - r_{n-1}).$$

In this form it is obvious that when $x = r_c$, every term vanishes except one, viz. that which does not contain the factor $x - r_c$.

Now let r_c and r_{c+1} be two adjacent real roots of $f(x) = 0$. Suppose x to vary from r_c continuously to r_{c+1}. The factors $r_c - r_i$ and $r_{c+1} - r_i$ have the same sign, since by hypothesis r_i is greater or less than either r_c or r_{c+1}. But $r_c - r_{c+1}$ and $r_{c+1} - r_c$ have different signs. Hence $(a, d) f_1(r_c)$ and $(a, d) f_1(r_{c+1})$ have different signs or the same sign according as r_c and r_{c+1} have the same sign or different signs. Hence $(a, d) f_1(x) = 0$ has an odd number of real roots between every two adjacent roots of $f(x) = 0$ except in the case of the least positive and the greatest negative root. In this case, $(a, d) f_1(x) = 0$ has an even number of roots or none in the interval.

If any root of $f(x) = 0$ is repeated N times, it is easily shown that the same root will be repeated $N - 1$ times in $(a, d) f_1(x) = 0$.

In order to see that this theorem includes Rolle's, it is only necessary to make $a = n$ and $d = -1$, and to observe that $(n, -1) f_1(x)$ divided by x produces the first derivative of the Differential Calculus, the function contemplated in Rolle's theorem.

Since 0 is a root of $(n, -1) f_1(x) = 0$, it follows from what was shown above that $f'(x) = 0$ has an odd number of roots between the least positive and greatest negative root of $f(x) = 0$.

Obviously

$$(a, d) f_1(x) = f(x) \left[a - d \left(\frac{r_1}{x - r_1} + \frac{r_2}{x - r_2} + \ . \ . \ . \ + \frac{r_n}{x - r_n} \right) \right].$$

When x increases *numerically* through a root of $f(x) = 0$, and $f(x)$ changes sign, the other factor of $(a, d) f_1(x)$ also changes sign, passing through ∞. If d is positive, this factor becomes negative; if d is negative, it becomes positive. Hence when $f(x)$ changes sign, it becomes unlike or like $(a, d) f_1(x)$ according as d is positive or negative; that is, the two-term series $f(x), (a, d) f_1(x)$ gains or loses one variation of sign according as d is positive or negative whenever $f(x)$ changes sign as x increases *numerically*.

The following are useful results of the foregoing theorem : —

1. In determining whether $f(x) = 0$ has equal roots, instead of $f(x)$ and $f'(x)$, we may employ $(0, 1) f_1(x)$ and $f'(x)$, with a slight saving of labor.

2. Since the n successive $(0, 1)$ derivatives of $f(x)$ have the essential properties of Fourier's functions, Fourier's theorem with slight modification is true of the former functions; and since the changes of sign lost or gained on account of imaginary roots, are lost or gained for different values of x, in general, in the two series of functions, the existence of imaginary roots may often be shown by the

application of Fourier's theorem to both series of functions.   An example will illustrate.

Let $\qquad\qquad f\ (x) = x^4 - x^3 + 2x^2 - 3x + 2$;

then $\qquad\qquad f'\ (x) = \qquad 4x^3 - 3x^2 + 4x - 3$,

$\qquad\qquad \tfrac{1}{2}f''\ (x) = \qquad\qquad 6x^2 - 3x + 2$,

$\qquad\qquad \tfrac{1}{3!}f'''(x) = \qquad\qquad\qquad 4x - 1$,

$\qquad\qquad \tfrac{1}{4!}f^{\mathrm{iv}}\ (x) = \qquad\qquad\qquad\qquad + 1$.

When $x = 0$, the series has four changes of sign, and when $x = 1$, it has no changes.   Since four changes are lost, the number of real roots between o and 1 is either four or less than four by an even number.

Again, let $\qquad f(x) = x^4 - x^3 + 2x^2 - 3x + 2$;

then $\qquad\qquad (0,1)f_1(x) = \qquad - x^3 + 4x^2 - 9x + 8$,

$\qquad\qquad \tfrac{1}{2}(0,1)f_2(x) = \qquad\qquad 2x^2 - 9x + 12$,

$\qquad\qquad \tfrac{1}{3!}(0,1)f_3(x) = \qquad\qquad\qquad - 3x + 8$,

$\qquad\qquad \tfrac{1}{4!}(0,1)f_4(x) = \qquad\qquad\qquad\qquad + 2$.

This series presents no change of sign either when $x = 0$ or when $x = 1$.   Hence $f(x) = 0$ has no roots between o and 1; this result taken with the former result shows that $f(x) = 0$ has no real roots.

3. The following theorem, which is a corollary of the foregoing, may sometimes be useful :—

The number of real roots of $f(x) = 0$ between a and $a + b$, a being either positive or negative, and b being positive, equals, or is less than by an even integer, the number of changes of sign in the series

$$\frac{1}{n!}f''(a)b^n + \frac{1}{(n-1)!}f^{n-1}(a)b^{n-1} + \frac{1}{(n-2)!}f^{n-2}(a)b^{n-2} + \ldots + \qquad f'(a)b + f(a),$$

$$\frac{1}{(n-1)!}f^{n-1}(a)b^{n-1} + \frac{2}{(n-2)!}f^{n-2}(a)b^{n-2} + \ldots + \frac{n-1}{1}f'(a)b + nf(a),$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$n!\,f(a).$$

Call the first member of this series $F(b)$.   The following members are the n successive $(0, 1)$ derivatives of $F(b)$.   Now $F(x) = 0$ is an equation whose roots are respectively less or greater by a than those of $f(x) = 0$ according as a is positive or negative.   Hence $F(x) = 0$ has the same number of roots between o and b that $f(x) = 0$ has between a and $a + b$.   When $x = 0$, the series $F(x)$, $(0, 1)$ $F_1(x)$, $(0, 1)_2 F(x)$, ... $(0, 1) F_n(x)$ presents no change of sign.   Hence by Four-

ier's theorem modified to correspond to the functions employed, the number of changes of sign in the series $F(b), (0, 1) F_1(b), \ldots (0, 1) F_n(b)$ equals or exceeds by an even integer the number of roots of $f(x) = 0$ lying between a and $a + b$.

In practice it is most convenient to take $b = 1$. Sometimes, however, it is necessary to assume either a or b fractional in order to distinguish two nearly equal real roots from two imaginary roots. We will illustrate the method by a single example.

$$
\begin{array}{lrrrrr}
\text{Let } f\ (x) = x^4 - 2x^3 + 5x^2 - & x + 3; & + 12 & + 36 & + 72 & + 72 \\
\text{then } \quad f'\ (x) = & 4x^3 - 6x^2 + 10x - 1, & - 3 & - 6 & - 6 \\
\tfrac{1}{2}f''\ (x) = & 6x^2 - 6x + 5, & + 10 & + 10 \\
\tfrac{1}{3!}f'''(x) = & 4x - 2, & - 2 \\
\tfrac{1}{4!}f^{iv}\ (x) = & 1. \\
\hline
& & + 6 \quad + 17 \quad + 40 \quad + 66 \quad + 72
\end{array}
$$

To determine the number of roots between 0 and 1, we make $a = 0$ and $b = 1$ and obtain the series $+ 6, + 17, + 40, + 66, + 72$. Hence there are no roots in this interval. But as Fourier's theorem applied to Fourier's functions places all the roots in this interval, it follows that the equation has no real roots.

GEOMETRIC DIVISION OF NON-CONGRUENT QUANTITIES.

By Prof. E. W. Hyde, Cincinnati, O.

Grassmann has treated the subject of division in the fourth chapter of his *Ausdehnungslehre* of 1844. He shows that the quotient of two of the quantities which he is treating is, in general, indefinite; but that, if the divisor and dividend are *congruent*, the quotient is definite, being simply the numerical ratio of their magnitudes. He has not, however, treated in general the ratio of two quantities of the same order, merely touching briefly, in the preface to the first edition, upon the ratio of two vectors. He says, in §141 of the same book: "*Der quotient stellt dann und nur dann einen einzigen, endlichen Werth dar, wenn der Divisor von geltendem Werthe ist, und zugleich entweder selbst als Grösse nullter Stufe dargestellt werden kann, oder dem Dividend gleichartig ist.*" By "*Werth*" he means an extensive quantity of some order, and with this meaning the statement is perfectly true; but I propose to show how a definite signification may be assigned to the quotient of two quantities of the same order, but not congruent, and also to consider the general quotient of two quantities, one of which is not a factor of the other.

We will take up first, however, one or two cases of quotients in which the divisor is a factor of the dividend, or *vice versa*.

The defining equation of division is

$$\frac{B}{.A}\, A = B, \quad \text{or} \quad A\frac{\vdots B}{A.} = B;$$

using the dot as Grassmann does to indicate whether A is to be the first or second factor, i. e. whether A operates on $\dfrac{B}{A}$, or $\dfrac{B}{A}$ on A. We have, if p_1, p_2, etc. are points,

$$\frac{p_1 p_2 p_3}{p_1 p_2.} = p_3 + x p_1 + y p_2;$$

for, multiplying both sides by $p_1 p_2$, we have the identity $p_1 p_2 p_3 = p_1 p_2 p_3$. As x and y may have any values whatever, the quotient is any point in the plane $p_1 p_2 p_3$; only, however, lying on the line $p_1 p_2$ when $x + y = \infty$. If $x + y = 0$, the quotient is $p_3 + x\,(p_1 - p_2)$, i.e. any point in a line through p_3 parallel to $p_1 p_2$. In plane space $p_1 p_2 p_3$ is a scalar quantity; therefore, dividing by it, we have the reciprocal of $p_1 p_2$, viz.:

$$\frac{1}{p_1 p_2} = \frac{1}{p_1 p_2 p_3}\,(p_3 + x p_1 + y p_2).$$

In this case, p_3 is any point not collinear with p_1 and p_2. Similarly,

$$\frac{1}{p_1} = \frac{1}{p_1 p_2 p_3} (p_2 p_3 + x p_3 p_1 + y p_1 p_2),$$

in which p_2 and p_3 are any two points not collinear with p_1. In solid space we have, similarly, $p_1 p_2 p_3 p_4$ being scalar, while $p_1 p_2 p_3$ is not so,

$$\frac{1}{p_1 \cdot} = \frac{1}{p_1 p_2 p_3 p_4} (p_2 p_3 p_4 + x p_3 p_4 p_1 + y p_4 p_1 p_2 + z p_1 p_2 p_3);$$

so that the reciprocal is any plane whatever, or more accurately, some *portion* of any plane whatever.

We will now consider quotients of quantities of the same order. Such a quotient will be itself of the *zero* order, i. e. in some sense a *numerical* quantity. If the divisor and dividend are congruent, the quotient is a mere number, viz. the ratio of their magnitudes. The ratios of points will first be discussed. We have $\frac{p_2}{\cdot p_1} p_1 = p_2 = \tau p_1$, say. Regarding $\frac{p_2}{\cdot p_1} (= \tau)$ as an operator, we see that its effect upon p_1 is to change it into p_2, i. e. to move p_1 a definite distance in a definite direction. τ is therefore a directed quantity, and yet numerical, as having no geometrical dimensions; we may call it a *directed number* in virtue of its character, and a *transferrer* in virtue of its office. Looking at the ratio of two points in this way, it appears that, although *directed*, it should yet, on account of its numerical nature, have no *positional* qualities; i. e., if $p_1 p_2$ and $p_3 p_4$ have the same direction and are equal in length, we may assume $\frac{p_2}{\cdot p_1} = \frac{p_4}{\cdot p_3}$, or $\tau_{12} = \tau_{34}$. Since we are to use these ratios as operators, we shall always write them to the left of the operand, and may omit the dot in the denominator. $\frac{p_2}{p_1} = \frac{p_4}{p_3}$ gives $p_2 = \frac{p_4}{p_3} p_1$, and $\frac{p_2}{p_1} p_3 = p_4$, but *not* $p_2 p_3 = p_1 p_4$; because $p_1 \frac{p_2}{p_1} p_3 = p_1 \left(\frac{p_2}{p_1} p_3 \right)$, i. e. the quantity in parentheses is to be multiplied as a *whole* by p_1.

If τ is an operator which moves *any* point a definite distance in a given direction, then evidently $\tau (\tau p) = \tau^2 p$ is a point twice as far from p as τp, in the same direction. Similarly, τ^n moves p n times as far as τ, and it is easy to see that the exponential law holds good as for ordinary numbers, whether n be $+$, $-$, or fractional. Since $\tau^{-n} (\tau^n p) = p$, it appears that, if the exponent of τ be negative, the transference is in the opposite direction. We have $\tau^{-1} \tau = \tau \tau^{-1} = \tau^0 = \frac{\tau}{\tau} = 1$. Also, by Fig. 1,

$$\tau_1^{-1} \tau_2 \tau_1 e = \tau_2 \tau_1^{-1} \tau_1 e = \frac{\tau_2}{\tau_1} \tau_1 e = \tau_2 e;$$

but
$$\frac{\tau_2 e}{\tau_1 e}\tau_1 e = \tau_2 e; \quad \therefore \quad \frac{\tau_2 e}{\tau_1 e} = \frac{\tau_2}{\tau_1}.$$

And again, $\tau_2^{-1}\tau_1^{-1} = (\tau_2\tau_1)^{-1} = \frac{1}{\tau_1\tau_2}$. Thus the τ's are subject to the ordinary numerical laws of multiplication and division among themselves.

Also, if e be any point, $\tau e - e = (\tau - 1)e = \varepsilon$, say, is a vector; hence $\tau - 1$ is an operator that changes a point into a vector, i. e. moves it to infinity and reduces its weight to zero.

$(\tau - 1)^{-1}(\tau - 1)e = e = (\tau - 1)^{-1}\varepsilon$; hence $(\tau - 1)^{-1}$ is an operator which changes a vector into a point.

$\tau'(\tau - 1)e = \tau'\tau e - \tau'e = \tau e - e = (\tau - 1)e$; hence the τ operator has no effect on a vector.

Of course $\tau_m - \tau_n$ is an operator of the same kind as $\tau - 1$.

By the figure,
$$\tau_1 + \tau_2 = 2\tau_1^{\frac{1}{2}}\tau_2^{\frac{1}{2}},$$
$$\therefore \tau_1^2 + 2\tau_1\tau_2 + \tau_2^2 = 4\tau_1\tau_2,$$
$$\therefore (\tau_1 - \tau_2)^2 = 0;$$

i. e. the operator $\tau_2 - \tau_1$ reduces a vector to zero.

We have, also, $\quad \tau^n e = e + n(\tau e - e) = (1 + n\tau - n)e;$
$$\therefore \tau^n = 1 - n + n\tau;$$

from which the numerical character of τ is apparent.

Since the sum of n unit points is n times the mean point of the system, we have

$$\tau_1 e + \tau_2 e + \ldots + \tau_n e = \overset{n}{\underset{1}{\Sigma}}\tau . e = n\bar{\tau}e; \quad \therefore \frac{1}{n}\overset{n}{\underset{1}{\Sigma}}\tau = \bar{\tau} \qquad (1)$$

is an operator which moves a point to the mean of the positions to which τ_1, τ_2, etc. move it.

Now if $\tau_1, \tau_2, \ldots \tau_n$ be successively applied to the same point, i. e. the point be multiplied by τ_1, this product by τ_2, etc., the final position of the point will be in the right line through the original point and the mean of all the points $\tau_1 e, \tau_2 e$, etc., and n times as far from the original point as the mean point is; hence

$$(\tau_n\tau_{n-1}\ldots\tau_1)^{\frac{1}{n}} = (\tau_1\tau_2\ldots\tau_n)^{\frac{1}{n}} = \bar{\tau} = \frac{1}{n}\overset{n}{\underset{1}{\Sigma}}\tau; \qquad (2)$$

i. e. the arithmetical and geometrical means of a series of τ-operators are equal.

It may be easily shown that we have also more generally

$$\frac{\overset{n}{\underset{1}{\Sigma}}(m\tau)}{\overset{n}{\underset{1}{\Sigma}}m} = (\tau_1^{m_1}\tau_2^{m_2}\ldots\tau_n^{m_n})^{\frac{1}{\Sigma m}}. \qquad (3)$$

Let ϵ be the base of Naperian logarithms, and write

$$\epsilon^\tau = 1 + \tau + \frac{1}{2!}\tau^2 + \frac{1}{3!}\tau^3 + \ldots, \tag{4}$$

in which the right hand member is to be the interpretation of the left. Then, by (3),

$$\epsilon^\tau = \left(2 + \frac{1}{2!} + \frac{1}{3!} + \ldots\right)\left(\tau^2\tau^{\frac{1}{2}}\tau_3!\tau_4! \ldots\right)^{\overline{2 + \frac{1}{2} + \frac{1}{3!} + \ldots}}^{\,1} = \epsilon\tau. \tag{5}$$

$$\therefore \log \epsilon\tau = 1 + \log \tau = \tau, \text{ or } \log \tau = \tau - 1. \tag{6}$$

Therefore $\log \tau$ is an operator that changes a point into a vector. From (5)

$$\tau = \epsilon^{-1}\epsilon^\tau = \epsilon^{\tau-1}. \tag{7}$$

Let us next consider the equation

$$\tau = \tau_1{}^x \tau_2{}^y \tau_3{}^z, \tag{8}$$

in which τ_1, τ_2, τ_3 are non-coplanar unit transferrers, x, y, z scalar variables, and τ a variable transferrer. If τ of equation (8) operate on any point e, it will transfer e to any point of space, when suitable values are given to x, y, z.

If one relation connect x, y, z, then τe will be a point on some surface; if two relations subsist between the scalar variables, the locus of τe is a curve. Suppose x, y, z to be all functions of t; by equation (6) equation (8) becomes

$$\log \tau = \tau - 1 = x \log \tau_1 + y \log \tau_2 + z \log \tau_3;$$

$$\therefore \frac{d\tau}{dt} = \frac{dx}{dt}\log \tau_1 + \frac{dy}{dt}\log \tau_2 + \frac{dz}{dt}\log \tau_3 = \log \tau_1{}^{\frac{dx}{dt}}\tau_2{}^{\frac{dy}{dt}}\tau_3{}^{\frac{dz}{dt}}. \tag{9}$$

By (6) $\frac{d\tau}{dt}$ changes e into a vector, and this vector is evidently parallel to the tangent at the point τe; hence for the equation of the tangent line, we have

$$\tau' = \tau + \frac{d\tau}{dt}w. \tag{10}$$

If x and y are independent, we have

$$\frac{d\tau}{dx} = \log \tau_1\tau_3{}^{\frac{dz}{dx}} \quad \text{and} \quad \frac{d\tau}{dy} = \log \tau_2\tau_3{}^{\frac{dz}{dy}}; \tag{11}$$

and for the tangent plane to a surface,

$$\tau' = \tau + \frac{d\tau}{dx}u + \frac{d\tau}{dy}v. \tag{12}$$

For example, the equation of the hyperbola may be written, omitting the e from both sides,

$$\tau = \tau_1' \tau_2'^{-1};$$ (13)

whence for the tangent

$$\tau' = \tau_1' \tau_2' \tau'^{-1} + \log \tau_1'' \tau_2^{-wt-2}$$
$$= \tau_1' \tau_2'^{-1} + \tau_1'' \tau_2^{-wt-2} - 1, \text{ by (6)},$$
$$= \tau_1'^{+w} \tau_2^{(t-w)t-2}, \text{ by (3)}.$$ (14)

Similarly for the helix, we have

$$\tau = \tau_1^{a \cos \theta} \tau_2^{a \sin \theta} \tau_3^{c\theta};$$ (15)

and for the tangent plane,

$$\tau' = \tau_1^{a} (\cos \theta - w \sin \theta) \tau_2^{a} (\sin \theta + w \cos \theta) \tau_3^{c} (\theta + w).$$ (16)

From the equation $\dfrac{\tau_2 - 1}{\tau_1 - 1} (\tau_1 - 1) e = (\tau_2 - 1) e$, it appears that $\dfrac{\tau_2 - 1}{\tau_1 - 1}$ is an operator which changes the vector $(\tau_1 - 1) e$ into the vector $(\tau_2 - 1) e$; i. e. turns the first vector through a certain angle and changes its length in the ratio of the lengths of τ_1 and τ_2. When the lengths of τ_1 and τ_2 are the same, we will write

$$\frac{\tau_2 - 1}{\tau_1 - 1} = v = \frac{\varepsilon_2}{\varepsilon_1},$$ (17)

if we make $\varepsilon_2 = (\tau_2 - 1) e$ and $\varepsilon_1 = (\tau_1 - 1) e$.

The expression in (17) is a *versor* for which reason we use the German v to represent it.

$\dfrac{\varepsilon}{e}$ is an operator which changes the point e into the vector ε, and if $\varepsilon = (\tau - 1) e$, we may write

$$\frac{\varepsilon}{e} = \tau - 1.$$ (18)

Similarly,

$$\frac{e}{\varepsilon} = (\tau - 1)^{-1}.$$ (19)

The ratio of two points has been treated by Unverzagt in his *Theorie der goniometrischen und longimetrischen Quaternionen*, but in a manner quite different from the above.

We will now consider the ratio of two vectors purely as a *versor*, not giving to such ratios the compound character of versor and vector as Hamilton did, whereby his calculus became necessarily one of complexes.

In the first place, in *plane* space, the versor is *scalar* in character, since rota-

tion can occur only in one plane, and therefore two rotations can differ only in magnitude and sign. In fact, v is in this case simply $(1 - 1)^x$.

In solid space, however, if $v = \dfrac{\varepsilon_2}{\varepsilon_1}$, and $T\varepsilon_2 = T\varepsilon_1$, v turns any vector parallel to the plane $\varepsilon_1\varepsilon_2$ through an angle equal to that from ε_1 to ε_2 and in the same direction; i. e. the rotation is about an axis perpendicular to the plane $\varepsilon_1\varepsilon_2$. Hence, in this case, v like τ, is a directed number. It is still essentially $(1 - 1)^x$, with a quality of direction added.

Let us consider the general effect of the operator v on any vector ε. Since v is purely a versor, it ought not to change in any way the *character* of ε; that is, $v\varepsilon$ remains a *vector*. Suppose ε is parallel to the axis of v; then, as neither ε nor v has particular *position*, they simply having a common point at infinity, and as v can produce no change of *direction* of ε about the axis of v to which ε is parallel, it appears that v should have *no effect whatever* upon ε; that is, when ε is parallel to the axis of v, $v\varepsilon = \varepsilon$.

Next suppose ε and v (Fig. 2) to make some angle between 0 and $90°$ with each other. Take ε' perpendicular to v and ε'' parallel to v, so that $\varepsilon = \varepsilon' + \varepsilon''$;

then $$v\varepsilon = v(\varepsilon' + \varepsilon'') = v\varepsilon' + v\varepsilon'',$$

assuming the distributive law. But we have just seen that

$$v\varepsilon'' = \varepsilon''; \quad \therefore \quad v\varepsilon = \varepsilon'' + v\varepsilon',$$

and v revolves ε *conically* through some angle θ about the axis of v. Thus v exactly corresponds to Hamilton's $q(\ )q^{-1}$.

We see thus to what we are led, if we preserve the idea of *geometric dimensions* by making the ratio of two vectors essentially numerical, and an operator distinct from a vector which is a geometric magnitude of the first order.

The question arises, how it is that Hamilton was led to make the assumption that the properties of versor and vector *must* be combined in the same quantity, as asserted in Article 64 of Tait's Quaternions. The gist of the whole matter is in this, that Hamilton assumed that *any given operator must always have the same meaning, or effect, independently of the operand;* something that is by no means axiomatic.

Let ι_1, ι_2, ι_3 (Fig. 3) be three mutually rectangular unit vectors, and i_1, i_2, i_3 three correspondingly parallel versors, as in the figure, so that

$$\frac{\iota_3}{\iota_2} = i_1, \quad \frac{\iota_1}{\iota_3} = i_2, \quad \frac{\iota_2}{\iota_1} = i_3.$$

Then because $-\iota_2 = i_1\iota_3 = i_1(i_1\iota_2) = i_1^2\iota_2$, Hamilton assumes that the square of a versor must always and everywhere be -1, without reference to what is operated upon. But, according to the natural interpretation we have already arrived at from

considering versors purely as such, for the case of a versor operating on a parallel vector, we should have $i_1^2 \iota_1 = i_1 (i_1 \iota_1) = i_1 \iota_1 = \iota_1$; that is, with ι_1 for operand, $i_1^2 = 1$ instead of $- 1$.

Again, consider the operator $i_1 i_2 i_3$; we have

$$i_1 i_2 i_3 \iota_2 = i_1 i_2 (- \iota_1) = i_1 \iota_3 = - \iota_2,$$

so that, in this case, the versor is equivalent to $- 1$, and its value is assumed by Hamilton to be therefore necessarily always $- 1$.   Take, however, ι_1 and ι_2 successively as operands, and we find

$$i_1 i_2 i_3 \iota_1 = i_1 i_2 \iota_2 = i_1 \iota_2 = \iota_3 = i_2^{-1} \iota_1; \ \therefore \ \text{in this case } i_1 i_2 i_3 = i_2^{-1};$$

$$i_1 i_2 i_3 \iota_3 = i_1 i_2 \iota_3 = i_1 \iota_1 = \iota_1 = i_2 \iota_3; \qquad \therefore \ \text{in this case } i_1 i_2 i_3 = i_2 .$$

In order to make an algebra in which $i_1^2 = i_1 i_2 i_3 = - 1$ invariably, it was unavoidably necessary to assume $i_1 = \iota_1$ etc., thus introducing complexity, and ignoring completely the useful and convenient idea of geometric dimensions.   The fact that Grassmann's whole system is founded upon this idea of dimensions, is that which renders it so much more natural, simple, and practical than quaternions.

Quotient of two lines (point-vectors) in plane space.   We have, if L_1 and L_2 are point-vectors ; i. e. each is the product of two points, or of a point and a vector, $\dfrac{L_2}{L_1} L_1 = L_2$; so that $\dfrac{L_2}{L_1}$ is an operator which changes L_1 into L_2.   Call such an operator Λ, so that $\Lambda L = L'$; the question arises how to determine L'; i. e. what is the effect of Λ on other lines besides the one that appears in the denominator of its value.   A natural interpretation of the meaning of ΛL may be arrived at as follows : —

Using the sign | as Grassmann does to signify *complement*, write $L = | \, p$, and then assume $\Lambda L = | \, \tau p$.   This will enable us to construct ΛL with ease.   We have at once $(\Lambda - 1) L = | \, (\tau - 1) p$; but $(\tau - 1) p$ is a point at ∞ in a definite direction, and hence $(\Lambda - 1) L$, its complement, must be a line through the mean point of the reference triangle.   Thus, as to every τ there corresponds a certain point at infinity in a definite direction from p, so to every Λ there corresponds a certain line through the mean of the reference points, cutting L in a definite point.

Since we have $(\tau^n - 1) p = n (\tau - 1) p$, we have also the complementary equation

$$(\Lambda^n - 1) L = n (\Lambda - 1) L, \tag{20}$$

by the aid of which, if we know ΛL, we can at once construct $\Lambda^n L$, as in Fig. 4. It is evident that all the relations between τ operators shown in equations (1) to (7) hold equally for Λ operators.

Fig. 5 is a diagram corresponding to Fig. 1 for points. ι_0 is the mean point of the reference system, the three points of this system not being shown, as they are

unnecessary; it corresponds to the line at infinity in the τ system.   To A_1 there belongs the line $(A_1 - 1) L$, coinciding with $c_0 c_1$; to A_2 there belongs, in the same way $c_0 c_2$; and to $A_1 A_2$ belongs $c_0 c_{12}$.   These correspond to the points at infinity in the directions of τ_1, τ_2, and $\tau_1 \tau_2$ of Fig. 1.

As in Fig. 1, $c, \tau_1 \tau_2 c, (\tau_1 \tau_2)^{\frac{1}{2}} c$ are collinear, so in Fig. 5, $L, A_1 A_2 L$, and $(A_1 A_2)^{-\frac{1}{2}} L$ pass through one point c_{12}.   As in Fig. 1, $(\tau_1 \tau_2)^{\frac{1}{2}} c = \frac{1}{2} (\tau_1 + \tau_2) c$, so in Fig. 5, $(A_1 A_2)^{-\frac{1}{2}} L = c_{12} d = \frac{1}{2} cf = \frac{1}{2} (A_1^{-1} + A_2^{-1}) L$.   To make the construction clearer, the following relations are given: $L = c_1 a = c_2 b = c_{12} c$; $A_1 A_2 L = lk = ih = c_{12} g$; $A_1 L = lm$; $A_2 L = in$.

Writing now the equation

$$A = A_1^x A_2^y, \qquad\qquad (21)$$

A is an operator which by giving suitable values to x and y will move L to any position whatever in the plane space under consideration.   If some relation subsist between x and y then A causes L to be always tangent to some curve.   If A, A_1, A_2, L correspond reciprocally to τ, τ_1, τ_2, c, then when $y = f(x)$, (21) may be called the line equation of the curve reciprocal to that represented by the point equation $\tau = \tau_1^x \tau_2^y$. Furthermore, $\dfrac{dA}{dx} = \log A_1 A_2^{\frac{dy}{dx}}$ is an operator which makes L pass through the point of contact of AL, and also through the mean of the reference points.   Hence

$$A' = A + z \frac{dA}{dx} \qquad\qquad (22)$$

is the equation of the point of contact of the line AL.

Thus $$A = A_1' A_2'^{-1} \qquad\qquad (23)$$

is the equation of the curve reciprocal to the hyperbola of equation (13), and of course represents different conics according to the position of the mean point with reference to that hyperbola.   The equation

$$A' = A_1^{t+z} A_2^{(t-z)t-z} \qquad\qquad (24)$$

represents the point of contact.   Since the line $A^\infty L$ coincides with the line $(A - 1) L$, and since in (23) the exponent of A_1 is ∞ when that of A_2 is 0, and *vice versa*, it appears that $(A_1 - 1) L$ and $(A_2 - 1) L$ are tangent to the curve enveloped by (23) at the points $LA_1 L$ and $LA_2 L$.   This corresponds to the statement that the points at infinity, $(\tau_1 - 1) c$ and $(\tau_2 - 1) c$, are the points of contact of the tangents, $c \tau_1 c$ and $c \tau_2 c$, to the curve represented by (13), i. e. of the asymptotes.

Quotient of two plane-vectors and of two point-plane-vectors in solid space. Let η_1 and η_2 be two plane vectors, i. e. each is the product of two line vectors. Then $\dfrac{\eta_2}{\eta_1}$ changes the plane-vector η_1 into η_2; i. e. revolves η_1 through a certain angle about some axis parallel to η_1 and η_2.   Thus the operation is identical with that

performed by the ratio of two vectors.   Hence we may use the same letter v to
designate this operator.   If v operate on a plane vector γ not parallel to its axis,
then γ may be resolved into two components, one parallel and one perpendicular
to the axis of v; the parallel component will be rotated through the angle of v,
while no effect can be produced on the other component; hence the total effect of
v on γ will be to rotate it conically about the axis of v through a definite angle.

Next let P_1 and P_2 be two planes or point-plane-vectors; i. e. each is the pro-
duct of three points.   Then

$$\frac{P_2}{P_1} = \Pi_{12} , \text{ say},\tag{25}$$

is an operator that changes P_1 into P_2.   Now suppose the Π operators to play the
same part in solid space that the A operators do in plane space; i. e. if $P = \mid p$,
then $\Pi P = \mid \tau p$.   Thus we have a reciprocal system in three dimensional space.

The equation $$\Pi = \Pi_1{}^x \Pi_2{}^y \Pi_3{}^z\tag{26}$$

may represent any plane whatever; i. e. Π will move a given plane P to any posi-
tion by giving suitable values to x, y, z.   If one relation subsist between x, y, z,
(26) will represent some convex or some skew surface, or else some curve.   If two
relations subsist between them, then (26) will represent a developable surface.   If
one of the scalars x, y, z be constant, and the others be connected by some equa-
tion, then (26) will represent a cone.

Of course, on the assumption above as to the nature of the Π operators, they
will satisfy the same equations of relation as have been proved for the τ operators.
To every Π corresponds a plane $(\Pi - 1) P$ through the mean point of the refer-
ence system and cutting P in a fixed line, and the construction of any particular
value of ΠP in (26) is easy theoretically, though the diagram would be pretty
complex.

We may now, by the aid of these operators, express the quotient of any two
geometric quantities.   Take, for instance, $\frac{p_1 p_2 p_3}{p_4}$.   If the points are all coplanar,
we have $p_1 p_2 p_3 = n p_1 p_2 p_4$, where n is some scalar multiplier.

Thus $$\frac{p_1 p_2 p_3}{p_4} = \frac{n p_1 p_2 p_4}{p_4} = n p_1 p_2 + x p_2 p_4 + y p_4 p_1.$$

If, however, the points are not coplanar, we may write

$$\frac{p_1 p_2 p_3}{\cdot p_4} = p_1 p_2 \tau_{43} + x p_2 p_4 + y p_4 p_1 + z p_3 p_4;$$

τ_{43} being the operator that transfers p_4 to p_3.   Again, take $\frac{p_3 p_4 p_5}{p_1 p_2}$.   If the points are

coplanar, we have

$$p_4 = m_1 p_1 + m_2 p_2 + m_3 p_3,$$

$$p_5 = n_1 p_1 + n_2 p_2 + n_3 p_3;$$

$$\therefore\ p_4 p_5 = \begin{vmatrix} p_2 p_3 & p_3 p_1 & p_1 p_2 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix},$$

and

$$p_3 p_4 p_5 = \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} p_1 p_2 p_3.$$

Hence the quotient becomes

$$\begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} \frac{p_1 p_2 p_3}{p_1 p_2} = \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} (p_3 + x p_1 + y p_2).$$

If the points are non-coplanar, we may write

$$p_5 = n_1 p_1 + n_2 p_2 + n_3 p_3 + n_4 p_4;$$

whence

$$p_3 p_4 p_5 = n_1 p_3 p_4 p_1 + n_2 p_3 p_4 p_2;$$

and the required quotient is

$$\frac{n_1 p_1 p_3 p_4 + n_2 p_2 p_3 p_4}{\cdot p_1 p_2} = p_4 (n_1 A_{12\cdot13} + n_2 A_{12\cdot23}) + x p_1 + y p_2.$$

In a similar way other cases may be treated.

ON THE INTERIOR CONSTITUTION OF THE EARTH AS RESPECTS DENSITY.

By Dr. G. W. Hill, Washington, D. C.

Nearly all the matter accessible to us is found to be porous. Thus the application of pressure to it tends to reduce the amount of porosity and, in consequence, augments the density of the mass. Moreover, the greater the pressure the greater is the increment of density. A familiar instance of this is the case of atmospheric air or a gas in which, provided the temperature remains constant, the density varies directly as the pressure.

It is natural to think that the matter of which the earth is composed is not excepted from this law. At small depths, it is true, the rigidity of the earth's mass interferes with its exerting any pressure, as the existence of caves shows. But at great depths where the weight of the superincumbent mass becomes very great, it is extremely probable the molecular force of cohesion gives way in a manner which allows pressure to act; which is illustrated by the behavior of ice in a glacier.

I propose to see what conclusions we are led to by adopting this relation between the density ρ and the pressure p,

$$\rho = A + Bp.$$

A and B are constants, A denoting the density at the surface, and B the rate of increase of the density per unit of pressure. In applying this formula to the atmosphere and gases, we have by Boyle's law $A = 0$. Let V denote the potential of the gravitating force of the whole mass, and let us neglect the effect of the centrifugal force arising from the rotation of the earth. Then pressure being supposed to act as though the whole mass were fluid, hydrostatics furnishes us with the equation

$$dp = \rho dV.$$

V being restricted to points on the surface or in the interior of the mass, it satisfies the partial differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + 4\pi\rho = 0.$$

The three equations now written may be regarded as determining the three unknowns ρ, p, and V.

By the elimination of V and p we get

$$\frac{\partial^2 \log \rho}{\partial x^2} + \frac{\partial^2 \log \rho}{\partial y^2} + \frac{\partial^2 \log \rho}{\partial z^2} + 4\pi B\rho = 0.$$

It will be seen that the constant A has disappeared from this equation.   By Boyle's law in the case of gases $A = 0$; that is, the matter is capable of attenuating itself to an infinite degree, a thing very improbable.   But the introduction of the constant term A, and consequent supposition of a limit to the attenuation, does not change the differential equation which ρ satisfies.   This partial differential equation contains the whole theory of gases under a uniform temperature contained in vessels of any figure, and acted on by any gravitating forces; also the theory of atmospheres surrounding solid nuclei of density as heterogeneous as we please, and of any figure.   The truth of the equation is not at all invalidated by any discontinuity in ρ or B; these quantities may change the law of their values as often as the problem demands.

The very simple integral of this equation in the case of the earth's atmosphere, when the attraction of the atmosphere on itself is neglected, is well known. It is our object here to examine the special solutions of this equation which are defined by the equation,

$$\rho = \text{function} \left[\sqrt{(x^2 + y^2 + z^2)} \right].$$

In this case, making $r = \sqrt{(x^2 + y^2 + z^2)}$, the partial differential equation is reduced to an ordinary one and becomes

$$\frac{d \cdot r^2 \dfrac{d \cdot \log \rho}{dr}}{dr} + 4\pi B r^2 \rho = 0,$$

or, as it may be written,

$$\frac{d^2 (r \log \rho)}{dr^2} + 4\pi B r \rho = 0.$$

To simplify this, let us put

$$s = 4\pi B r^2 \rho.$$

Then s being made the dependent variable, we have

$$\frac{d \cdot r^2 \dfrac{d \cdot \log s}{dr}}{dr} + s - 2 = 0.$$

And if $\log r = v$, it becomes

$$\frac{d^2 \log s}{dv^2} + \frac{d \cdot \log s}{dv} + s - 2 = 0.$$

Furthermore, if $\dfrac{d \cdot \log s}{dv} = u$, this differential equation of the first order between

u and s is obtained

$$\frac{du}{ds} = \frac{2 - (u + s)}{us}.$$

This being integrated, and u obtained in terms of s, or s in terms of u, r is given by the equation

$$r = K \varepsilon^{\int \frac{ds}{us}},$$

or by the equation

$$r = K \varepsilon^{\int \frac{du}{2 - (u + s)}},$$

in which K is an arbitrary constant. And, if in the first of these values of r, $4\pi B r^2 \rho$ is substituted for s, the equation will be obtained which determines ρ as a function of r.

The differential equation in u and s is a particular case of the general form

$$Pdx + Qdy = 0,$$

where P and Q denote algebraical functions of x and y of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F.$$

Mathematicians have been able to obtain the integral of this, in finite terms, only when the constants A, B, etc. satisfy certain equations of condition.* Unfortunately, the differential equation under consideration does not belong to any of these particular cases. Recourse must be had to series or other methods of approximation for the determination of the relation between u and s. However, the differential equation itself will furnish the properties of the family of plane curves it defines.

Thus u and s denoting the rectangular co-ordinates of a point in a plane, the differential equation gives immediately the means of drawing the tangent to the curve which passes through this point. Excepting at the two singular points whose co-ordinates are $u = 0$, $s = 2$ and $u = 2$, $s = 0$, for which the expression of the tangent takes the indeterminate form

$$\frac{du}{ds} = \frac{0}{0},$$

the system of curves do not intersect each other, since there is but one value of $\frac{du}{ds}$ for given values of u and s. Since the differential equation is satisfied by the condition $s = 0$, the axis of u is itself one of the system of curves, and no curve can cross it except at the point $u = 2$. If, in the differential equation, we substi-

*See Liouville, *Journal de Mathématiques*, 2e Series, Tom. III, p. 417.

tute $2 + du$ for u, and ds for s, it is clear that only one curve passes through this
point, and that its tangent here is given
by the equation $du/ds = -\frac{1}{3}$. The
axis of u, between the points $u = 2$
and $u = \infty$, is an asymptote to the
whole system of curves. The axis of
s is intersected at right angles by the
system of curves. Investigating what
occurs at the point $s = 2$ on this axis,
we substitute du for u and $2 + ds$ for
s, and obtain for determining du/ds at
this point the following quadratic

$$\left(\frac{du}{ds}\right)^2 + \frac{1}{2}\frac{du}{ds} + \frac{1}{2} = 0,$$

the roots of which are imaginary.
Hence no curve passes through this
point, and it is easy to see that the system of curves makes an infinite number of
turns about it.

The tangent to any curve, at its intersection with the straight line whose
equation is $u + s = 2$, is parallel to the axis of s. When u and s are both very
great, the tangent to the curve approximates to parallelism with the axis of s.
When s is very great and u small in comparison, the differential equation becomes
approximately

$$u\frac{du}{ds} = -1;$$

or integrated, $$u^2 = 2(s_0 - s),$$

if s_0 is the value of s when $u = 0$. Hence the curves in the vicinity of the axis
of s approximate to the parabola, in measure as we recede from the origin of co-
ordinates.

It is very easy to draw the curves connecting all the points possessing parallel
tangents. For convenience let a denote the common value of ds/du for these
points; then the differential equation furnishes

$$(u + a)(s + a) = a(a + 2).$$

Thus these curves are equilateral hyperbolas having their asymptotes parallel to
the axis of co-ordinates.

Thus much in regard to the properties of the curves defined by the differ-

ential equation under consideration. But, for the special physical problem we have in view, there is no necessity to attend to the course of the curves through the whole plane. The density being supposed to increase with augmentation of pressure, B is necessarily positive, and r and ρ, from the nature of the problem, being the same; s is likewise a positive quantity. There is then need only of considering the curves on the positive side of the axis of u. Moreover, since
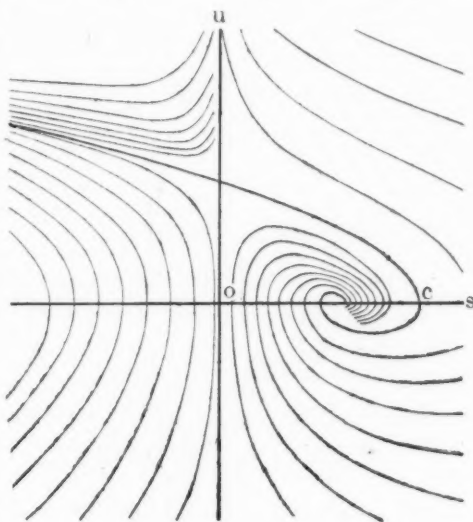
$$u = \frac{d \cdot \log (r^2 \rho)}{d \cdot \log r} = \frac{r}{\rho} \frac{d\rho}{dr} + 2,$$

and $d\rho/dr$ is always negative when the force is directed towards the centre of the mass, there is no need of attending to the curves in the portion of the plane for which $u > 2$.

Before proceeding to the special problem we have in hand, I propose to illustrate the general theory by considering the density of the earth's atmosphere. It must be remembered that, in the usual manner of treating this question, the attraction of the atmosphere on itself is neglected; here, however, it is taken into account. Boyle's law being supposed to hold exactly, we shall have

$$\rho = Bp.$$

To integrate the differential equation between u and s, it will be necessary to obtain from observation the initial values of these two variables which hold at the surface of the earth. Let us denote these by u_0 and s_0; and by a similar notation the values of all the variables at the earth's surface. The values of u_0 and s_0 result from those of certain well-known physical constants.

Let
$D =$ the density of mercury,
$h =$ the altitude of the barometer,
$g =$ the force of gravity,
$R =$ the mean density of the earth.

From an equation just given we have

$$u_0 = r_0 \left(\frac{d \cdot \log \rho}{dr} \right)_0 + 2$$

$$= \frac{r_0}{p_0} \left(\frac{dp}{dr} \right)_0 + 2.$$

But we also evidently have

$$p_0 = gDh,$$

$$\left(\frac{dp}{dr} \right)_0 = - g\rho_0.$$

Substituting these values, $\qquad u_0 = 2 - \dfrac{\rho_0 r_0}{Dh}.$

Thus it is apparent that u is independent of the units assumed for the measurement of lengths and densities.   In the next place

$$B = \frac{\rho_0}{p_0} = \frac{\rho_0}{gDh}.$$

But we have

$$g = \frac{4\pi R r_0^3}{3} \cdot \frac{1}{r_0^2} = \tfrac{4}{3}\pi R r_0.$$

Thence we get

$$s_0 = 4\pi B r_0^2 \rho_0 = \frac{3\rho_0^2 r_0}{DRh}.$$

Thus s is also independent of the just mentioned units.

Let us adopt the following values of the constants which enter into the expressions of u_0 and s_0:—

$$r_0 = 6365419 \text{ metres,}$$
$$h = 0.76 \text{ metres,}$$
$$\rho_0 = 0.001293187,$$
$$D = 13.596,$$
$$R = 5.67.$$

The value of ρ_0 is that found by Regnault* for the temperature $0°$ of the centigrade scale and the given altitude of the barometer; r_0 is the distance of his observatory from the centre of the earth according to Bessel's dimensions of the terrestrial spheroid; and the value of R is that determined by Baily in his repetition of the Cavendish experiment.   With these data we obtain the following values of u_0 and s_0:—

$$u_0 = -794.6425,$$
$$s_0 = 0.5450835.$$

Having these initial values we can easily integrate the differential equation connecting u and s by mechanical quadratures or series, in the direction of s diminishing until s becomes so small as to be of no account.   The corresponding values of r and ρ could then be found as we have already explained.   However, the differences between the numerical values obtained by this method and those resulting from neglecting the action of the atmosphere on itself would be insensible.

We pass now to the problem of the mass of the earth.   Let us here denote the values of the variables which hold at the centre by the subscript ($_0$).   If the

*Mémoires de l'Académie des Sciences de Paris, Tom. XXI.

density at the centre be finite we must have $s_0 = 0$; and the differential equation

$$\frac{ds}{du} = \frac{us}{2 - (u + s)}$$

shows that $u_0 = 2$, else s would be 0 for all values of u. Hence the curve we have to consider, in this case, is the single one which passes through the singular point $u = 2$, $s = 0$.

The mass included in the sphere whose radius is r, is

$$M = \frac{1}{B} \int_0^r sdr$$

$$= -\frac{1}{B} r^2 \frac{d . \log \rho}{dr}$$

$$= \frac{1}{B} r (2 - u).$$

Hence, denoting the values of the variables at the earth's surface by the subscript $(_1)$, and R denoting, as before, the mean density of the earth, we shall have

$$\frac{4\pi}{3} Rr_1{}^3 = \frac{r_1 (2 - u_1)}{B} .$$

Whence we derive
$$B = \frac{3 (2 - u_1)}{4\pi R r_1{}^2},$$

and
$$s_1 = 3 (2 - u_1) \frac{\rho_1}{R}.$$

Then if we draw in the plane the right line whose equation is

$$s = 3 \frac{\rho_1}{R} (2 - u),$$

the co-ordinates of its intersection with the curve defined by the differential equation and passing through the singular point $u = 2$, $s = 0$, will be the values of u_1 and s_1. This right line passes through the point $u = 2$, $s = 0$, and it is readily ascertained from the differential equation that upon this curve u constantly diminishes as s augments until it becomes 0. The lines can therefore intersect on the positive side of the axis of s only when

$$6 \frac{\rho_1}{R} > OC,$$

where OC is the distance from the origin of the point where the mentioned curve crosses the axis of s.

In order to illustrate the general theory by an application, I have computed

by mechanical quadratures the values of the variable s and the function necessary for obtaining r. For this purpose it will be well to substitute for the independent variable u the variable $z = 2 - u$. The results obtained are given in the following table at intervals of 0.1 in z:—

z	s	s/z	$\int \frac{dz}{z-z}$	$\log r$	$\log s/r^2$
0.0	0.000	3.000	$-\infty$	$-\infty$	0.4771
0.1	0.294	2.940	$-$ 1.1360	9.5065	0.4553
0.2	0.576	2.879	$-$ 0.7737	9.6640	0.4323
0.3	0.846	2.818	$-$ 0.5546	9.7592	0.4088
0.4	1.103	2.757	$-$ 0.3938	9.8290	0.3845
0.5	1.348	2.695	$-$ 0.2646	9.8851	0.3594
0.6	1.580	2.633	$-$ 0.1551	9.9326	0.3333
0.7	1.799	2.570	$-$ 0.0589	9.9744	0.3061
0.8	2.005	2.507	$+$ 0.0279	0.0121	0.2780
0.9	2.198	2.442	$+$ 0.1078	0.0468	0.2485
1.0	2.378	2.378	$+$ 0.1825	0.0792	0.2176
1.1	2.543	2.312	$+$ 0.2533	0.1100	0.1854
1.2	2.695	2.246	$+$ 0.3213	0.1396	0.1514
1.3	2.832	2.178	$+$ 0.3874	0.1682	0.1155
1.4	2.953	2.110	$+$ 0.4522	0.1964	0.0776
1.5	3.060	2.040	$+$ 0.5163	0.2242	0.0372
1.6	3.149	1.968	$+$ 0.5806	0.2522	9.9939
1.7	3.222	1.895	$+$ 0.6457	0.2804	9.9473
1.8	3.276	1.820	$+$ 0.7123	0.3094	9.8966
1.9	3.310	1.742	$+$ 0.7816	0.3394	9.8414
2.0	3.322	1.661	$+$ 0.8547	0.3712	9.7791
2.1	3.309	1.576	$+$ 0.9336	0.4055	9.7088
2.2	3.265	1.484	$+$ 1.0215	0.4436	9.6266
2.3	3.182	1 384	$+$ 1.1239	0.4881	9.5365

Let us suppose that the surface density of the earth $\rho_1 = 2.7$ and the mean density $R = 5.67$. Then at the surface of the earth the value of s/z must be

$$\frac{s_1}{z_1} = 3 \frac{\rho_1}{R} = 1.4286.$$

By interpolating in the table it is found that this value corresponds to the following values of the principal variables:—

$$z = 2.257,$$
$$s = 3.224,$$
$$\log r = 0.4681,$$
$$\log \frac{s}{r^2} = 9.5722.$$

Now the last two quantities are the logarithms of the surface values of the radius and the density measured in such units as in every case will give the simplest values to the arbitrary constants. But let us take the radius at the surface as the linear unit, and represent the surface density as 2.7. Then to reduce the numbers so as to correspond to these units, it is evident we must add 9.5319 to the logarithms in the column of $\log r$, and 0.8592 to the logarithms in the column of $\log s/r^2$. Thus are obtained the following corresponding values of r and ρ:—

r	ρ	r	ρ
0.000	21.69	0.469	10.25
0.109	20.63	0.501	9.43
0.157	19.57	0.535	8.65
0.195	18.54	0.570	7.88
0.230	17.53	0.608	7.13
0.261	16.54	0.649	6.40
0.291	15.58	0.694	5.70
0.321	14.63	0.743	5.02
0.350	13.72	0.800	4.35
0.379	12.81	0.866	3.70
0.408	11.93	0.945	3.06
0.438	11.08	1.000	2.70

It will be noticed that the density at the centre is almost double of that given

by Laplace's formula; and it seems that this supposition as to the law of density will not fit the phenomena as well as the latter.

The limit beneath which the ratio ρ_1/R cannot be reduced without the problem failing to have a solution, is of interest.   If the curve employed for the solution of this problem is prolonged until its tangent passes through the singular point on the axis of u, which it plainly must do before the curve crosses the axis of s a second time, this tangent affords the limit sought for the ratio $3\,\rho_1/R$.   The tangents of the curves, at the points of the plane whose co-ordinates satisfy the equation

$$\frac{2-(u+s)}{us} = \frac{u-2}{s},$$

pass through the mentioned singular point.   This equation in a simpler form is

$$s = (1+u)(2-u),$$

which consequently represents a parabola passing through both singular points, and having its axis parallel to that of s.   By the employment of mechanical quadratures, the following additional points of the curve have been obtained:—

s	z	s	z
3.0	2.420	2.3	2.499
2.9	2.458	2.2	2.478
2.8	2.486	2.1	2.446
2.7	2.505	2.0	2.403
2.6	2.515	1.9	2.345
2.5	2.518	1.8	2.264
2.4	2.513	1.75	2.204

From these it is evident the point $u = -0.2$, $s = 1.76$ which lies on the just-mentioned parabola is also very nearly on the employed curve.   Hence if ρ_1/R is less than a fraction which is approximately $\frac{4}{15}$, there is no solution.

The number of solutions in any particular case is deserving of notice.   The integral

$$\int \frac{dz}{s-z}$$

is proportional to the value of $\log r$.   It does not become infinite until the curve has made an infinite number of turns about the singular point on the axis of s.

This may be shown by a transformation of variables. Let us adopt polar co-ordinates, the singular point being the pole, and thus put

$$s = w \cos \theta + 2,$$
$$z = w \sin \theta + 2.$$

The differential equation then becomes

$$\frac{dw}{w} = - \frac{w \sin \theta \cos^2 \theta + \sin^2 \theta + \sin \theta \cos \theta}{w \cos \theta \sin^2 \theta + 1 + \sin^2 \theta - \sin \theta \cos \theta} \, d\theta.$$

And we have

$$\int \frac{dz}{s - z} = \int \frac{d\theta}{w \cos \theta \sin^2 \theta + 1 + \sin^2 \theta - \sin \theta \cos \theta}.$$

The denominator of these expressions cannot vanish unless w exceed 2, and it is plain that it remains positive and finite for all values of θ. Thus r becomes infinite only when θ does. Consequently there are an infinite number of solutions when $\rho_1/R = \frac{1}{3}$; and a less number when ρ_1/R is either less or greater than this. With the value we have attributed to this fraction in the case of the earth, the course of the curve shows that there is but one solution.

EXERCISES.

164

DERIVE geometrically the usual expressions for the radius of the circle inscribed in a triangle, and for the area of the triangle, in terms of the sides.

[*F. H. Loud.*]

165

IN any triangle ABC let a circle be inscribed touching the sides AB, BC, CA in N, L, M respectively. Let the centre O of this circle be joined to the vertices, and from O let OP, OQ be drawn perpendicular respectively to OC, OB, and cutting BC in P and Q. Then if NP and AQ be drawn, these lines will be parallel as will also AP and MQ. [*F. H. Loud.*]

166

A CIRCLE cuts a parabola and the centroid of the four points of intersection is found. What is the locus of the centre of the circle if this point be fixed?

[*W. M. Thornton.*]

167

If $y_1 y_2 = k^2$ (a constant), where (x_1, y_1) and (x_2, y_2) are points on the parabola $y^2 = 4ax$, find the locus of the intersections of the normals at these points.

[*R. H. Graves.*]

168

If the normals at four points on a rectangular hyperbola meet in a point, and the sum of the squares on the six distances between the four points, taken two together, is constant ($= k^2$), prove that the locus of the point of concourse of the normals is a circle.   [*R. H. Graves.*]

169

Find the locus of the instantaneous centre of a tangent to an ellipse when one point of the tangent moves in the auxiliary circle.   [*R. H. Graves.*]

170

Prove that the circular projections on planes passing through one extremity of the transverse axis of an ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, of all sections through that axis, lie on a surface which is formed by eliminating φ between the two equations

$$(x - a \sin^2 \varphi)^2 + (y - ac\lambda \sin \varphi)^2 + (z - ab\varkappa \sin \varphi)^2 = a^2 \cos^2 \varphi,$$

$$x \sin \varphi + c\lambda y + b\varkappa z = a \sin \varphi;$$

in which
$$\lambda = \left(\frac{b^2 - a^2 \cos^2 \varphi}{b^2 - c^2} \right)^{\frac{1}{2}},$$

and
$$\varkappa = \left(\frac{a^2 \cos^2 \varphi - c^2}{b^2 - c^2} \right)^{\frac{1}{2}}.$$

[*J. O'Byrne Croke.*]

171

A homogeneous heavy rod is hung from a fixed point by elastic threads of given length fastened at its extremities.   Find the position of equilibrium.

[*W. M. Thornton.*]

172

Find the deflection of a homogeneous elastic beam of length $2v$, loaded uniformly, and supported at two points distant u from its middle point.

[*W. M. Thornton.*]

173

Find the deflection of a cross tie in a railway under the action of the driving

wheels of a locomotive, assuming the resistance of the road-bed to compression to be proportional to the deflection at the point considered.   [*W. M. Thornton.*]

SELECTED.

174

FIND the locus of the intersection of the altitudes of a triangle whose base and area are given.

175

FIND the locus of the poles of normals to a given ellipse.

176

FIND the normal to an ellipse which cuts the curve again at the minimum angle.

177

Two conjugate diameters a', b' subtend at a point on the ellipse angles a', β'. Show that $\cot^2 a' + \cot^2 \beta'$ is constant.

178

FIND the point in which the normal to $xy = m^2$ cuts the curve again.

179

FIND the area of the triangle formed by the asymptotes to an equilateral hyperbola and the normal to the curve.

180

FIND the envelope of a system of circles each of which is seen from two fixed points under a constant angle.

181

SHOW that the locus of the centres of equilateral hyperbolas circumscribed to a given triangle is the nine-points circle of the triangle.

182

THE centroid of the four points of intersection of a circle and an equilateral hyperbola, bisects the join of their centres.

183

IF $z = x + iy$ be a complex quantity whose geometric locus is a straight line, find the locus of $\zeta = z^2$.

184

SHOW how to express the area between the parabola $y = A + Bx + Cx^2$, the axes of co-ordinates, and the ordinate $x = 1$, by means of two given ordinates.

185

FROM a point taken on a fixed normal to a given ellipse the three other normals to the curve are drawn and the circumcircle of their feet is constructed. Find the locus of its centre.

186

FIND the locus of points which may be the common centre of two similar conics, one circumscribed, the other inscribed to a given triangle.

187

ON the diagonals of a complete quadrilateral three pairs of points are taken which divide these diagonals harmonically. Show that they form a Pascal's hexagon.

188

THREE conics are bitangent each to the other two. Show that the chords of contact of two of them with the third, form with the common tangents to these two an harmonic pencil.

189

SHOW how to resolve a given force into three coplanar components acting in given lines not concurrent.

190

D, E, F are the feet of the altitudes of ABC. Find the resultant of the force represented by AE, AF, BF, BD, CD, CE.

191

A ROD whose centroid is given, is hung from a smooth pin by a string fastened to its ends. Find the positions of equilibrium.

192

A FRUSTUM of a right circular cone stands on a rough inclined plane in a position of tottering equilibrium. Will it slide or topple over?

193

WHEN will the centroid of a triangular frame of wire coincide with that of three particles P, Q, R at its angular points?

194

SHOW that the centroid of a hemispheroid of very small eccentricity e deviates from the middle points of its altitude by $\frac{23}{12} e^2$ thereof, nearly.
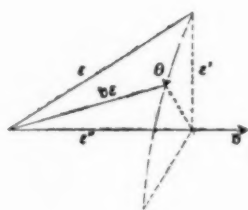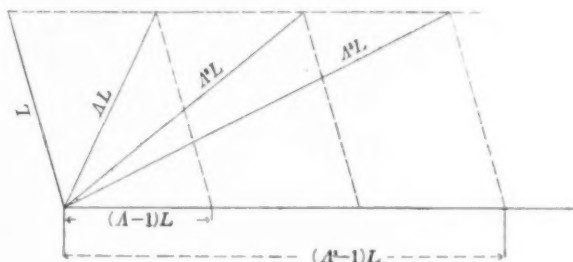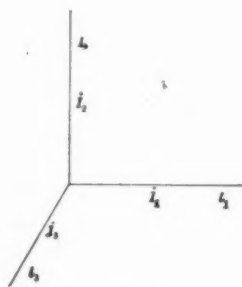
Fig. 1.

Fig. 2.

Fig. 4.

Fig. 3.

Fig. 5.

CONTENTS.

PAGE.

PUBLICATIONS RECEIVED.

Report of the Director of the Dearborn Observatory. 1885–86.

Observations made at the U. S. Naval Observatory. 1883.

Physical Arithmetic. By Alex. Macfarlane. London: Macmillan & Co.

Annales de l'Observatoire de Nice.

Curves of Pursuit. By Jas. M. Ingalls. (Analyst.)

Report of the Superintendent of the U. S. Naval Observatory for the year ending
June 30, 1887.

Rendiconti del Circolo Matematico di Palermo. Tom. I.

The Planetary and Stellar Worlds. By O. M. Mitchel. New York: Jno. B.
Alden.

Essai d'une Nouvelle Théorie Élémentaire des Logarithmes. By V. Jamet.
Paris: Nony & Cie.

Curso de Analyse Infinitesimal. By F. Gomes Teixeira. Porto: Typographia
Occidental.

Ephemerides for Physical Observations of Jupiter, 1888. By A. Marth. (Monthly
Notices of the Royal Astronomical Society.)

Publications of the Lick Observatory of the University of California, Vol. I. By
E. S. Holden.

Forty-Second Annual Report of the Director of Harvard College Observatory.

Astronomical Observations and Researches made at Dunsink, the Observatory
of Trinity College, Dublin. Sixth Part. By A. A. Rambaut.